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Analytic Continuation of the Harmonic Sums for the 3–Loop Anomalous Dimensions

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Abstract

We present for numerical use the analytic continuations to complex arguments of those basic Mellin transforms, which build the harmonic sums contributing to the 3–loop anomalous dimensions. Eight new basic functions contribute in addition to the analytic continuations for the 2–loop massless Wilson coefficients calculated previously. The representations derived have a relative accuracy of better than 10^{-7} in the range $x \, \epsilon \, [10^{-6}, 0.98]$.

1 Introduction

A very efficient way to calculate the evolution of the parton densities in deeply inelastic scattering consist in solving the evolution equations in Mellin–N space [1,2]. With the advent of the 3–loop anomalous dimensions [3, 4] the scaling violations of the deep–inelastic structure functions can be analyzed at high precision. The evolution equations are ordinary differential equations, which can be solved analytically. The Wilson coefficients [5–8] and anomalous dimensions are given in terms of multiple nested harmonic sums [9,10], which obey algebraic relations [9–12]. Each finite harmonic sum can be represented in terms of a Mellin transform of a harmonic polylogarithm [13] weighted by $1/(1 \pm x)$ and a polynomial of harmonic sums of lower weight. It turns out that for all basic functions at least up to weight five, as needed for the physical observables mentioned above, the harmonic polylogarithms are Nielsen integrals [14,15]. Furthermore, structural relations exist between various Mellin transforms of weighted Nielsen integrals [16]. They are due to relations at rational values of the Mellin variable, differentiation w.r.t. the Mellin variable, integration by parts relations and functional identities of Nielsen integrals. These relations, together with the quasi-shuffle algebra of harmonic sums allow to build the massless single–scale quantities in perturbative QED and QCD out of a low number of basic functions at a given order in perturbation theory.

The anomalous dimensions and Wilson coefficients in Mellin space, for instance, can be written in terms of these basic functions and a number of their derivatives, in addition to polynomials including Euler's ψ–functions as well as rational functions of the Mellin variable for complex values of N. Then, on this basis, results in x–space, are obtained by means of a single inverse Mellin transform, which has to be carried out numerically. For the class of functions contributing to the 2–loop Wilson coefficients the analytic continuations were given in Ref. [17]. Simple analytic continuations of nested sums are known for long, cf. e.g. Ref. [18]. This direction has been followed in Ref. [19] recently. However, in many cases the numerical convergence is very slow.

To obtain an appropriate representation for the 3–loop anomalous dimensions and 2–loop Wilson coefficients, we exploit the algebraic and structural relations between the nested harmonic sums. For the 2–loop anomalous dimensions only one non–trivial function contributes aside from Euler's $\psi^{(k)}(N)$ functions [20,21]. All known massless 2–loop Wilson coefficients can be expressed by four more functions [22, 23]. For the 3–loop anomalous dimension the Mellin transforms

$$\mathbf{M}[f(x)](N) = \int_0^1 dx\ x^N\ f(x) \tag{1.1}$$

of the following eight functions form the set of additional basic functions :

$$\frac{\mathrm{Li}_4(x)}{1 \pm x}, \qquad \frac{S_{1,3}(x)}{1+x}, \qquad \frac{S_{2,2}(x)}{1 \pm x}, \qquad \frac{\mathrm{Li}_2^2(x)}{1+x}, \qquad \frac{S_{2,2}(-x) - \mathrm{Li}_2^2(-x)/2}{1 \pm x}\ . \tag{1.2}$$

Here the functions $\mathrm{Li}_n(x)$ and $S_{p,n}(x)$ denote polylogarithms [24] and Nielsen–integrals [14, 15], respectively. In terms of harmonic polylogarithms [13], they read

$$\begin{aligned} &\mathrm{Li}_4(x) = H_{0,0,0,1}(x), \qquad S_{1,3}(x) = H_{0,1,1,1}(x), \qquad S_{2,2}(x) = H_{0,0,1,1}(x), \\ &\mathrm{Li}_2^2(x) = (H_{0,1}(x))^2, \quad S_{2,2}(-x) = H_{0,0,-1,-1}(x), \quad \mathrm{Li}_2^2(-x) = (-H_{0,-1}(x))^2, \end{aligned} \tag{1.3}$$

where the algebra can be applied to products of harmonic polylogarithms to express them in the basis of single harmonic polylogarithms.

In the present paper we derive fast and precise numerical representations of these Mellin transforms for complex values of N. Previously numerical representations of the 3–loop anomalous dimensions were given in [3,4]. [1] Following earlier investigations [17], we aim at numerical representations for the individual basic functions, the building blocks of the massless Wilson coefficients and the anomalous dimensions, to high precision. Thus, in principle, the present approach is tunable to arbitrary accuracy.

2 Mellin Transforms of the Type $f(x)/(1+x)$

As outlined in Ref. [17] the Mellin transforms containing the denominator $1/(1+x)$ are given by:

$$\begin{aligned}\mathbf{M}\left[\frac{f(x)}{1+x}\right](N) &= \ln(2)\cdot f(1) - \int_0^1 dx\; x^{N-1}\; \ln(1+x)\left[Nf(x) + xf'(x)\right] \\ &= \ln(2)\cdot f(1) - \sum_{k=1}^{20} a_k^{(1)}\left(N\,\mathbf{M}[f(x)](N+k-1) + \mathbf{M}\left[f'(x)\right](N+k)\right)\,.\end{aligned} \tag{2.1}$$

Accurate representations for the function $\ln(1+x)$ can be given using the minimax method for numerical approximation:

$$\ln(1+x) \simeq \sum_{k=1}^{20} a_k^{(1)} x^k \tag{2.2}$$

which we further improved to an accuracy of 3×10^{-16}. The coefficients read:

$$\begin{array}{lcllcl}
a_1^{(1)} & = & 0.9999999999999925\mathtt{D}{-}0 & a_2^{(1)} & = & -0.4999999999988568\mathtt{D}{-}0 \\
a_3^{(1)} & = & 0.3333333332641123\mathtt{D}{-}0 & a_4^{(1)} & = & -0.2499999977763199\mathtt{D}{-}0 \\
a_5^{(1)} & = & 0.1999999561535526\mathtt{D}{-}0 & a_6^{(1)} & = & -0.1666660875051348\mathtt{D}{-}0 \\
a_7^{(1)} & = & 0.1428517138099479\mathtt{D}{-}0 & a_8^{(1)} & = & -0.1249623936313475\mathtt{D}{-}0 \\
a_9^{(1)} & = & 0.1109128496887138\mathtt{D}{-}0 & a_{10}^{(1)} & = & -0.9918652787800788\mathtt{D}{-}1 \\
a_{11}^{(1)} & = & 0.8826572954250856\mathtt{D}{-}1 & a_{12}^{(1)} & = & -0.7643209265133132\mathtt{D}{-}1 \\
a_{13}^{(1)} & = & 0.6225829212455825\mathtt{D}{-}1 & a_{14}^{(1)} & = & -0.4572477090315515\mathtt{D}{-}1 \\
a_{15}^{(1)} & = & 0.2890194939889559\mathtt{D}{-}1 & a_{16}^{(1)} & = & -0.1496621145891488\mathtt{D}{-}1 \\
a_{17}^{(1)} & = & 0.6003156359511387\mathtt{D}{-}2 & a_{18}^{(1)} & = & -0.1731328252868496\mathtt{D}{-}2 \\
a_{19}^{(1)} & = & 0.3172112728405899\mathtt{D}{-}3 & a_{20}^{(1)} & = & -0.2760099875146713\mathtt{D}{-}4
\end{array}$$

For later use we also parameterize $\ln^2(1+x)$

$$\ln^2(1+x) \simeq \sum_{k=2}^{24} a_k^{(2)} x^k\ , \tag{2.3}$$

[1] Related earlier approximate representations were given in [25].

with

$$
\begin{array}{lclllcl}
a_2^{(2)} & = & 1.0000000000000000\mathtt{D}{-}0 & & a_3^{(2)} & = & -0.9999999999999985\mathtt{D}{-}0 \\
a_4^{(2)} & = & 0.9166666666663948\mathtt{D}{-}0 & & a_5^{(2)} & = & -0.8333333333136118\mathtt{D}{-}0 \\
a_6^{(2)} & = & 0.7611111103508889\mathtt{D}{-}0 & & a_7^{(2)} & = & -0.6999999819735105\mathtt{D}{-}0 \\
a_8^{(2)} & = & 0.6482139985629993\mathtt{D}{-}0 & & a_9^{(2)} & = & -0.6039649964806160\mathtt{D}{-}0 \\
a_{10}^{(2)} & = & 0.5657662306410356\mathtt{D}{-}0 & & a_{11}^{(2)} & = & -0.5323631718571445\mathtt{D}{-}0 \\
a_{12}^{(2)} & = & 0.5024238774786239\mathtt{D}{-}0 & & a_{13}^{(2)} & = & -0.4738508288315496\mathtt{D}{-}0 \\
a_{14}^{(2)} & = & 0.4427472719775835\mathtt{D}{-}0 & & a_{15}^{(2)} & = & -0.4029142806330511\mathtt{D}{-}0 \\
a_{16}^{(2)} & = & 0.3476841543351489\mathtt{D}{-}0 & & a_{17}^{(2)} & = & -0.2748590021353420\mathtt{D}{-}0 \\
a_{18}^{(2)} & = & 0.1915627642585285\mathtt{D}{-}0 & & a_{19}^{(2)} & = & -0.1130763066428224\mathtt{D}{-}0 \\
a_{20}^{(2)} & = & 0.5415661067306229\mathtt{D}{-}1 & & a_{21}^{(2)} & = & -0.1999877298940919\mathtt{D}{-}1 \\
a_{22}^{(2)} & = & 0.5303624439388411\mathtt{D}{-}2 & & a_{23}^{(2)} & = & -0.8944156375768203\mathtt{D}{-}3 \\
a_{24}^{(2)} & = & 0.7179502917974332\mathtt{D}{-}4 & & & &
\end{array}
$$

The following Mellin transforms contribute, cf. also Refs. [9, 10, 17],

$$\mathbf{M}\left[\mathrm{Li}_4(x)\right](N-1) = \frac{\zeta_4}{N} - \frac{\zeta_3}{N^2} + \frac{\zeta_2}{N^3} - \frac{S_1(N)}{N^4} \tag{2.4}$$

$$\mathbf{M}\left[\mathrm{Li}_4'(x)\right](N-1) = \mathbf{M}\left[\mathrm{Li}_3(x)\right](N-2) \tag{2.5}$$

$$\mathbf{M}\left[S_{1,3}(x)\right](N-1) = \frac{\zeta_4}{N} - \frac{(S_1(N))^3}{6N^2} - \frac{S_1(N)S_2(N)}{2N^2} - \frac{S_3(N)}{3N^2} \tag{2.6}$$

$$\mathbf{M}\left[S_{1,3}'(x)\right](N-1) = -\frac{1}{6}\mathbf{M}\left[\ln^3(1-x)\right](N-2) \tag{2.7}$$

$$\begin{aligned}\mathbf{M}\left[\mathrm{Li}_2^2(x)\right](N-1) &= \frac{1}{N}\left({\zeta_2}^2 - \frac{4\zeta_3}{N} - 2\zeta_2\frac{S_1(N)}{N} + 2\frac{S_{2,1}(N)}{N}\right) \\ &\quad + \frac{2}{N^3}\left((S_1(N))^2 + S_2(N)\right)\end{aligned} \tag{2.8}$$

$$\mathbf{M}\left[(\mathrm{Li}_2^2(x))'\right](N-1) = -2\,\mathbf{M}\left[\ln(1-x)\mathrm{Li}_2(x)\right](N-2) \tag{2.9}$$

$$\mathbf{M}\left[S_{2,2}(x)\right](N-1) = \frac{\zeta_4}{4N} - \frac{\zeta_3}{N^2} + \frac{(S_1(N))^2 + S_2(N)}{2N^3} \tag{2.10}$$

$$\mathbf{M}\left[S_{2,2}'(x)\right](N-1) = \mathbf{M}\left[S_{1,2}(x)\right](N-2) \tag{2.11}$$

$$\begin{aligned}\mathbf{M}\left[S_{2,2}(-x) - \mathrm{Li}_2^2(-x)/2\right](N-1) &= \frac{1}{N}\left[-\frac{3}{4}{\zeta_2}^2 + 2\mathrm{Li}_4(1/2) + \frac{7}{4}\zeta_3\ln(2) - \frac{1}{2}\zeta_2(\ln(2))^2\right. \\ &\quad \left. + \frac{1}{12}(\ln(2))^4 - \mathbf{M}\left[S_{1,2}(-x)\right](N-1)\right] \\ &\quad - \frac{1}{N}\left[\frac{{\zeta_2}^2}{8} + \mathbf{M}\left[\ln(1+x)\mathrm{Li}_2(-x)\right](N-1)\right]\end{aligned} \tag{2.12}$$

$$\begin{aligned}\mathbf{M}\left[\left(S_{2,2}(-x) - \mathrm{Li}_2^2(-x)/2\right)'\right](N-1) &= \mathbf{M}\left[S_{1,2}(-x)\right](N-2) \\ &\quad + \mathbf{M}\left[\ln(1+x)\mathrm{Li}_2(-x)\right](N-2)\end{aligned} \tag{2.13}$$

with

$$\mathbf{M}\left[\mathrm{Li}_3(x)\right](N-1) = \frac{\zeta_3}{N} - \frac{\zeta_2}{N^2} + \frac{S_1(N)}{N^3} \tag{2.14}$$

$$\mathbf{M}\left[\ln^3(1-x)\right](N-1) = -\frac{(S_1(N))^3}{N} - 3\frac{S_1(N)S_2(N)}{N} - 2\frac{S_3(N)}{N} \tag{2.15}$$

$$\mathbf{M}\left[S_{1,2}(x)\right](N-1) = \frac{\zeta_3}{N} - \frac{(S_1(N))^2}{2N^2} - \frac{S_2(N)}{2N^2} \tag{2.16}$$

$$\begin{aligned}\mathbf{M}\left[S_{1,2}(-x)\right](N-1) &= \frac{\zeta_3}{8N} - \frac{(\ln(2))^2}{2N^2} - \frac{(-1)^N}{N^2}\Big[S_{1,-1}(N) + \ln(2)\left(S_1(N) - S_{-1}(N)\right) \\ &\quad -\frac{1}{2}(\ln(2))^2\Big] \\ &\simeq \frac{\zeta_3}{8N} - \frac{1}{2N}\sum_{k=2}^{24}\frac{a_k^{(2)}}{N+k}\end{aligned} \tag{2.17}$$

$$\mathbf{M}\left[\ln(1-x)\mathrm{Li}_2(x)\right](N-1) = \frac{1}{N}\left[-2\zeta_3 - \zeta_2 S_1(N) + \frac{1}{N}\left((S_1(N))^2 + S_2(N)\right) + S_{2,1}(N)\right] \tag{2.18}$$

$$\mathbf{M}\left[\ln(1+x)\mathrm{Li}_2(-x)\right](N-1) \simeq \sum_{k=1}^{20} a_k^{(1)}\,\mathbf{M}\left[\mathrm{Li}_2(-x)\right](N-1+k) \tag{2.19}$$

$$\mathbf{M}\left[\mathrm{Li}_2(-x)\right](N-1) = -\frac{\zeta_2}{2N} + \frac{\ln(2)}{N^2} - \frac{(-1)^N}{N^2}\left(S_{-1}(N) + \ln(2)\right) \tag{2.20}$$

The functions $S_{1,-1}(N)$ and $S_{2,1}(N)$ have been parameterized in Ref. [17]

$$S_{1,-1}(N) = (-1)^{N+1}\,\mathbf{M}\left[\frac{\ln(1+x)}{1+x}\right](N) - \ln(2)\left(S_1(N) - S_{-1}(N)\right) + \frac{1}{2}(\ln(2))^2 \tag{2.21}$$

$$S_{2,1}(N) = \mathbf{M}\left[\left(\frac{\mathrm{Li}_2(x)}{1-x}\right)_+\right](N) + \zeta_2 S_1(N)\,, \tag{2.22}$$

where

$$\mathbf{M}\left[\frac{\ln(1+x)}{1+x}\right](N) \simeq \frac{1}{2}(\ln(2))^2 - \frac{N}{2}\sum_{k=2}^{24}\frac{a_k^{(2)}}{N+k}\ . \tag{2.23}$$

3 Mellin Transforms of the Type $\left(f(x)/(1-x)\right)_+$

Three functions of this type contribute. In case the numerator function is analytic at $x=1$ one may represent the functions of this class using the minimax method directly. Both $\mathrm{Li}_4(x)$ and $S_{2,2}(x)$ have branch points at $x=1$. Therefore we derive an analytic representation $\hat{f}_i(x)$ for $\left[f_i(x) - f_i(1)\right]/(1-x)$ valid for the region $x \to 1$ and determine the remainder part using the minimax method:

$$\frac{f_i(x) - f_i(1)}{1-x} = \hat{f}_i(x) + \sum_{k=0}^{k_{\max}} c_k^{(i)} x^k\ . \tag{3.1}$$

For $[\mathrm{Li}_4(x)-\zeta_4]/(1-x)$ one obtains

$$\begin{aligned}\hat{f}_1(x) &= -\zeta_3+\frac{1}{2}(\zeta_2-\zeta_3)(1-x)+\left(\frac{1}{6}\ln(1-x)+\frac{\zeta_2}{2}-\frac{11}{36}-\frac{\zeta_3}{3}\right)(1-x)^2\\ &\quad+\left(\frac{1}{4}\ln(1-x)+\frac{11}{24}\zeta_2-\frac{19}{48}-\frac{\zeta_3}{4}\right)(1-x)^3\\ &\quad+\left(\frac{7}{24}\ln(1-x)+\frac{5}{12}\zeta_2-\frac{\zeta_3}{5}-\frac{599}{1440}\right)(1-x)^4\\ &\quad+\left(\frac{5}{16}\ln(1-x)+\frac{137}{360}\zeta_2-\frac{79}{192}-\frac{\zeta_3}{6}\right)(1-x)^5 \qquad (3.2)\end{aligned}$$

$$\begin{array}{lcrlcr}
c_0^{(1)} & = & -3.187045493829754\mathtt{D}{-1} & c_1^{(1)} & = & 1.752102582962004\mathtt{D}{-0}\\
c_2^{(1)} & = & -3.926780960319761\mathtt{D}{-0} & c_3^{(1)} & = & 4.533622455411171\mathtt{D}{-0}\\
c_4^{(1)} & = & -2.764070067739643\mathtt{D}{-0} & c_5^{(1)} & = & 7.705635337822153\mathtt{D}{-1}\\
c_6^{(1)} & = & -1.412597571664758\mathtt{D}{-2} & c_7^{(1)} & = & -1.024735147728843\mathtt{D}{-1}\\
c_8^{(1)} & = & 2.072912823276118\mathtt{D}{-1} & c_9^{(1)} & = & -3.094485142894180\mathtt{D}{-1}\\
c_{10}^{(1)} & = & 3.100508803799690\mathtt{D}{-1} & c_{11}^{(1)} & = & -2.017510797419543\mathtt{D}{-1}\\
c_{12}^{(1)} & = & 7.682650942255444\mathtt{D}{-2} & c_{13}^{(1)} & = & -1.310258217741916\mathtt{D}{-2}
\end{array}$$

with $k_{\max}=13$. The Mellin transform of $\hat{f}_i(x)$ is given by a linear combination of

$$\mathbf{M}\left[(1-x)^M\right](N) = B(N+1,M+1) \qquad (3.3)$$

$$\mathbf{M}\left[(1-x)^M\ln(1-x)\right](N) = \left[\psi(M+1)-\psi(M+N+2)\right]B(N+1,M+1) \qquad (3.4)$$

$$\begin{aligned}\mathbf{M}\left[(1-x)^M\ln^2(1-x)\right](N) &= \Big\{\left[\psi'(M+1)-\psi'(M+N+2)\right]\\ &\quad+\left[\psi(M+1)-\psi(M+N+2)\right]^2\Big\}B(N+1,M+1)\,. \qquad (3.5)\end{aligned}$$

$[S_{2,2}(x)-\zeta_4/4)]/(1-x)$ obeys the representation (3.1), where

$$\begin{aligned}\hat{f}_2(x) &= (1-x)\Bigg[\left(-\frac{7}{40}x^5+\frac{767}{720}x^4-\frac{979}{360}x^3+\frac{899}{240}x^2-\frac{1069}{360}x+\frac{469}{360}\right)\ln(1-x)^2\\ &\quad+\left(\frac{947}{3600}x^5-\frac{11683}{7200}x^4+\frac{15221}{3600}x^3-\frac{4827}{800}x^2+\frac{18511}{3600}x-\frac{409}{150}\right)\ln(1-x)\\ &\quad-\frac{104641}{504000}x^5+\frac{11675141}{9072000}x^4-\frac{15334867}{4536000}x^3+\frac{14820287}{3024000}x^2\\ &\quad-\frac{19680697}{4536000}x+\frac{2964583}{1134000}\Bigg]\\ &\quad+\left(-\frac{1}{7}x^6+\frac{43}{42}x^5-\frac{667}{210}x^4+\frac{2341}{420}x^3-\frac{853}{140}x^2+\frac{617}{140}x-\frac{363}{140}\right)\zeta_3 \qquad (3.6)\end{aligned}$$

$$
\begin{array}{lcrlcr}
c_0^{(2)} & = & 2.319102959447070\mathtt{D}{-1} & c_1^{(2)} & = & -1.341835247346528\mathtt{D}{-0} \\
c_2^{(2)} & = & 3.141213505948200\mathtt{D}{-0} & c_3^{(2)} & = & -3.689298402805891\mathtt{D}{-0} \\
c_4^{(2)} & = & 2.066708069184852\mathtt{D}{-0} & c_5^{(2)} & = & -0.221726622455410\mathtt{D}{-0} \\
c_6^{(2)} & = & -2.866874606436715\mathtt{D}{-1} & c_7^{(2)} & = & 1.886633904893633\mathtt{D}{-1} \\
c_8^{(2)} & = & -5.104536639955758\mathtt{D}{-1} & c_9^{(2)} & = & 1.814194435712243\mathtt{D}{-0} \\
c_{10}^{(2)} & = & -4.810364434281610\mathtt{D}{-0} & c_{11}^{(2)} & = & 9.670512224938253\mathtt{D}{-0} \\
c_{12}^{(2)} & = & -14.74512233105934\mathtt{D}{-0} & c_{13}^{(2)} & = & 16.93126634317140\mathtt{D}{-0} \\
c_{14}^{(2)} & = & -14.39776586765419\mathtt{D}{-0} & c_{15}^{(2)} & = & 8.789666564310812\mathtt{D}{-0} \\
c_{16}^{(2)} & = & -3.642212170527524\mathtt{D}{-0} & c_{17}^{(2)} & = & 9.173661876074224\mathtt{D}{-1} \\
c_{18}^{(2)} & = & -1.060348165292345\mathtt{D}{-1} & & &
\end{array}
$$

and $k_{\max} = 18$.

The Nielsen integrals $S_{n,p}(x)$ are analytic at $x = -1$. Therefore $\left[S_{2,2}(-x) - \mathrm{Li}_2^2(-x)/2 - c_1\right] / (1-x)$ can be represented by a polynomial directly, where

$$c_1 = -\frac{7}{8}{\zeta_2}^2 + 2\mathrm{Li}_4(1/2) + \frac{7}{4}\zeta_3 \ln(2) - \frac{1}{2}\zeta_2(\ln(2))^2 + \frac{1}{12}(\ln(2))^4\,. \qquad (3.7)$$

We will do this for the functions $\left[S_{2,2}(-x) - c_1 - {\zeta_2}^2/8\right]/(1-x)$ and $\left[\mathrm{Li}_2^2(-x) - {\zeta_2}^2/4\right]/(1-x)$ separately.

The following representation for $\mathbf{M}[(S_{2,2}(-x) - c_1 - {\zeta_2}^2/8)/(1-x)]$ is obtained:

$$\mathbf{M}\left[\frac{S_{2,2}(-x) - c_1 - {\zeta_2}^2/8}{1-x}\right](N) = \sum_{k=0}^{23} \frac{b_k^{(1)}}{N+k+1}\ , \qquad (3.8)$$

and

$$
\begin{array}{lcrlcr}
b_0^{(1)} & = & -0.8778567156865530\mathtt{D}{-1} & b_1^{(1)} & = & -0.8778567156865530\mathtt{D}{-1} \\
b_2^{(1)} & = & 0.3721432843134470\mathtt{D}{-1} & b_3^{(1)} & = & -0.1834122712421077\mathtt{D}{-1} \\
b_4^{(1)} & = & 0.1030460620911406\mathtt{D}{-1} & b_5^{(1)} & = & -0.6362060457158173\mathtt{D}{-2} \\
b_6^{(1)} & = & 0.4208927186604172\mathtt{D}{-2} & b_7^{(1)} & = & -0.2933929772309200\mathtt{D}{-2} \\
b_8^{(1)} & = & 0.2130242091464232\mathtt{D}{-2} & b_9^{(1)} & = & -0.1597936899156854\mathtt{D}{-2} \\
b_{10}^{(1)} & = & 0.1230894254048324\mathtt{D}{-2} & b_{11}^{(1)} & = & -0.9689535635431825\mathtt{D}{-3} \\
b_{12}^{(1)} & = & 0.7755737888131504\mathtt{D}{-3} & b_{13}^{(1)} & = & -0.6263517402742744\mathtt{D}{-3} \\
b_{14}^{(1)} & = & 0.5031055861991529\mathtt{D}{-3} & b_{15}^{(1)} & = & -0.3922594818742940\mathtt{D}{-3} \\
b_{16}^{(1)} & = & 0.2868111320615437\mathtt{D}{-3} & b_{17}^{(1)} & = & -0.1887234737089442\mathtt{D}{-3} \\
b_{18}^{(1)} & = & 0.1068980760727356\mathtt{D}{-3} & b_{19}^{(1)} & = & -0.4971730708906830\mathtt{D}{-4} \\
b_{20}^{(1)} & = & 0.1797845625225957\mathtt{D}{-4} & b_{21}^{(1)} & = & -0.4695889540721375\mathtt{D}{-5} \\
b_{22}^{(1)} & = & 0.7830613264154134\mathtt{D}{-6} & b_{23}^{(1)} & = & -0.6232207394074941\mathtt{D}{-7}
\end{array}
$$

The corresponding polynomial in x has an accuracy of 2×10^{-15} in $[0,1]$.

Similarly, for $\mathbf{M}[(\mathrm{Li}_2^2(-x) - {\zeta_2}^2/4)/(1-x)](N)$ one obtains:

$$\mathbf{M}\left[\frac{\mathrm{Li}_2^2(-x) - {\zeta_2}^2/4}{1-x}\right](N) = \sum_{k=0}^{11} \frac{b_k^{(2)}}{N+k+1}\ , \qquad (3.9)$$

with

$$
\begin{array}{llrllr}
b_0^{(2)} & = & -0.6764520210934552\mathrm{D}{-}0 & b_1^{(2)} & = & -0.6764520137562308\mathrm{D}{-}0 \\
b_2^{(2)} & = & 0.3235476094265664\mathrm{D}{-}0 & b_3^{(2)} & = & -0.1764446743143206\mathrm{D}{-}0 \\
b_4^{(2)} & = & 0.1081940672246993\mathrm{D}{-}0 & b_5^{(2)} & = & -0.7181309059958118\mathrm{D}{-}1 \\
b_6^{(2)} & = & 0.4940999469881481\mathrm{D}{-}1 & b_7^{(2)} & = & -0.3290941711692155\mathrm{D}{-}1 \\
b_8^{(2)} & = & 0.1916664887064280\mathrm{D}{-}1 & b_9^{(2)} & = & -0.8589741767655388\mathrm{D}{-}2 \\
b_{10}^{(2)} & = & 0.2508898780543465\mathrm{D}{-}2 & b_{11}^{(2)} & = & -0.3476710199486832\mathrm{D}{-}3
\end{array}
$$

The corresponding polynomial in x has an accuracy of 2.5×10^{-11} for $x \,\epsilon\, [0,1]$.

In Table 1 the values of the maximum relative errors are given for the first 40 Mellin moments. Depending on the function accuracies better than $9 \cdot 10^{-10}$ to $8 \cdot 10^{-13}$ are obtained. We also compared the representation of the basic functions through numerical inversion of the Mellin moments by a complex contour integral. In the range $x \,\epsilon\, [10^{-6}, 0.98]$ maximal relative errors of 2 to 8 $\cdot$ 10^{-8} were obtained, irrespective of the value of the basic functions. In some regions in x even up to several order of magnitude better results were obtained.

	Moments, $N \leq 40$	Inversion
$[\mathrm{Li}_4(x) - \zeta_4]/(1-x)$	4.0D–12	5.1D–8
$[S_{2,2}(x) - \zeta_4/4]/(1-x)$	8.2D–10	5.4D–8
$[S_{2,2}(-x) - c_1]/(1-x)$	1.8D–10	4.5D–8
$[\mathrm{Li}_2^2(-x) - \zeta_2^2/4]/(1-x)$	3.0D–10	4.3D–8
$\mathrm{Li}_4(x)/(1+x)$	1.2D–12	8.0D–8
$S_{1,3}(x)/(1+x)$	2.2D–12	3.3D–8
$S_{2,2}(x)/(1+x)$	1.1D–12	1.8D–8
$\mathrm{Li}_2^2(x)/(1+x)$	2.2D–12	3.0D–8
$[S_{2,2}(-x) - \mathrm{Li}_2^2(-x)/2]/(1+x)$	7.2D–13	3.2D–8

Table 1: Relative accuracies of the representations.

4 Conclusions

Single scale quantities in massless QED and QCD such as the Wilson coefficients and anomalous dimensions can be represented by nested harmonic sums. Their representation may be reduced to a small set of basic functions in addition to Euler's $\psi^{(k)}(N)$ functions, each of the basic functions corresponding to one particular single harmonic sums. This is possible due to algebraic and structural relations between the harmonic sums. Up to the level of the 3–loop anomalous dimensions 13 such basic functions occur. We have derived fast and precise numerical representations for the eight new functions contributing at the level of the 3–loop anomalous dimensions in QCD. In the region $x \,\epsilon\, [10^{-6}, 0.98]$ which is physically most interesting their relative accuracy is better than 10^{-7}. If needed, the procedure outlined in the present paper can readily be generalized to

a higher level of accuracy at the expense of more coefficients in the representation. Due to use of polynomial forms in the analytic continuations very fast representations are obtained well suited for the use in QCD evolution codes. The basic functions discussed in the present paper form a part of the building blocks for the 3–loop coefficient functions and other higher order single scale quantities.

`FORTRAN` routines are available from the authors upon request.

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